# The Ancient Roots of Linear Algebra: An Analysis of Rectangular Arrays in Chinese Mathematics

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Apr/06/2024

**Abstract**

The paper recounts and revises the development of linear albegra system from ancient Chinese referenced on the acient Chinese text: *The Nine Chapters on the Mathematical Art*. The two main related Chapter that will be examined are: "Excess and Deficits" and "The Fangcheng Procedure" in both their mathematical concepts and some representitive questions. Furthermore, this paper will compare mordern approaches on linear algebra system to the ancient Chinese strategies, finding potential insight that reveals the process of how mathematics evolved.

# 1 Introduction

## 1.1 Excess and Deficits:

The "Excess and Deficits" issues can be seen as equivalent to solving a system of two equations with two unknowns, serving as the precursor to the fangcheng problems. Rather than immediately applying the methods, the chapter categorizes the problems into three main types: "Excess and Deficits," "Two Excess and two Deficits," and "Excesses with exactness or Deficits with exactness." The goal of this chapter is to elucidate and analyze the general methodologies for these three kinds of problems, contrasting the historical approaches with those from a modern perspective. [**3**] Furthermore, in the realm of ancient Chinese mathematics, when there was a challenge in conceptualizing or using mathematical symbols abstractly, a technique known as "The Rule of False Position" was developed [**6**] (mirroring a method also documented by ancient Egyptians) to solve these problems. In algebra, "The Rule of Double False Position" refers to the process of solving a linear equation by starting with two incorrect assumptions for the variable: one overly high and the other overly low. These initial guesses are then used to compute a more accurate value that falls between them. This article will go through the basic "Excess and Deficits" procedure as an guide, building up solid foundations for *The Fangcheng Procedure*.

## 1.2 The FangCheng Procedure:

The *FangCheng procedure*, as delineated in *The Nine Chapters on the Mathematical Art*, highlights several distinct aspects: First, the *FangCheng* problems are set out in two dimensions on the *counting board*, facilitating a visual and structured approach to problem solving. Second, the *FangCheng* method involves a unique process of elimination that resemble "*Gaussian elimination*"

and ”*termed bian cheng*. This process entails ”*cross-multiply*” an entire column by a single entry to simplify the equations. Lastly, the *FangCheng* method employs a distinctive strategy for back substitution that diverges from the conventional methods in modern linear algebra, offering an alternative perspective on solving systems of equations [**6**].
This article will first introduce the matrix representation from ancient Chinese mathematicans, then examining the *FangCheng procedure* with respect to positive and negative integers, finding out how ancient Chinese mathematician entails negative numbers within column subtractions. Morever, this article will briefly introduce the *elimination* of matrix, knowing how *back substitution* works and analyze the fangcheng procedure's *integer preserving property.*

# 2 Preliminaries

## 2.1 The Counting Board

Ancient Chinese mathematicians were adept at utilizing the counting board, an essential tool for their mathematical explorations and calculations. Known as the *suanpan*, it typically consisted of a wooden frame with rods, onto which beads were strung. This device facilitated a visual and tactile approach to arithmetic operations, enabling the execution of simple tasks like addition and subtraction, as well as more complex algorithms for multiplication and division, and even solving linear equations. This article will elucidate the basic procedure employed by ancient Chinese mathematicians in using the counting board to perform operations such as *"plus"*, *"minus"*, *"Multiplication"*, *"division"* [**2**]. The universal law for the counting board dictated that calculations should commence from left to right, with an empty slot representing zero. This methodological approach underscores the organized and systematic nature of ancient Chinese mathematical practices.

### 2.1.1 Addition & Subtraction

**Addition:**

| | | |
|---|---|---|
| 3 | 4 | 7 |
| 3 | 4 | 7 |
| 6 | **9*** | 4 |

Since $7 + 7 = 14$, we will have to "carry ove" 1 from unit place to the tens place, hence shifting $8 \to 9$.
**Subtraction:** Enforcing subtraction is similar to addition on the counting board, but the carrying happened at the colmn row.

| | | | | |
|---|---|---|---|---|
| 2 | 1 | 8 | 6 | 7 |
| 1 | 6 | 3 | 8 | 5 |
| $\cancel{1}$ | 5 | 4 | 8 | 2 |

At the thousandths place, we have carry the 1 left in ten-thousandths place. Hence, we shift the subtraction from $1 - 6$ to $11 - 6 = 5$.

### 2.1.2 Multiplication

Multiplication is relatively more complex than addition and subtraction. Consider a counting board multiply 495 with 612

| | | | | | |
|---|---|---|---|---|---|
| | | | 4 | 9 | 5 |
| | | | | | |
| | 6 | 1 | 2 | | |

The method is going to examined the product between 612 and the "hunderdth place", "unit place", and "tens place" number from 495 (i.e. Going from left to right). Every time we finish a round of multiplication we shifts 612 right with 1 column and erase the number we've multiplied from 495:

| | | | | | |
|---|---|---|---|---|---|
| | | | | 9 | 5 |
| 2 | 4 | 4 | 8 | | |
| | | 6 | 1 | 2 | |

$\Rightarrow$

| | | | | | |
|---|---|---|---|---|---|
| | | | | | 5 |
| 2 | 4 | 4 | 8 | | |
| | 5 | 5 | 0 | 8 | |
| | | 6 | 1 | 2 | |

$\Rightarrow$

| | | | | | |
|---|---|---|---|---|---|
| | | | | | |
| 2 | 4 | 4 | 8 | | |
| | 5 | 5 | 0 | 8 | |
| | | 3 | 0 | 6 | 0 |

$= 302940$

#### 2.1.3 Division

For division, it is similar with multiplication. It can deal with both "divisible" and "indivisible" cases by introducing fractional parts in the counting board. We won't get into the "indivisible" calculation and only consider the divisible in this artile. Consider the following counting board divide 56889 by 147:

| | | | | |
|---|---|---|---|---|
| | | 3 | | |
| 5 | 6 | 8 | 8 | 9 |
| 1 | 4 | 7 | | |

$\Rightarrow$

| | | | | |
|---|---|---|---|---|
| | | 3 | 8 | |
| 1 | 2 | 7 | 8 | 9 |
| | 1 | 4 | 7 | |

$\Rightarrow$

| | | | | |
|---|---|---|---|---|
| | | 3 | 8 | 7 |
| 0 | 1 | 0 | 2 | 9 |
| | | 1 | 4 | 7 |

$= 387$

From board 1 to board 2, $56889 - 14700 \times 3 = 12789$. From board 2 to board 3, $12789 - 1470 \times 8 = 1029$, lastly $1029 - 147 \times 7 = 0$. Each steps we minus the the greatest possible product for each number place, moving from left to the right.

In general, the designed counting calculation is extremely similar to the "Column Method" in modern mathematics [**7**], providing an efficient and rigorous way for precise calculations.

# 3 Chapter 7: Excess and Deficits:

In this chapter, the absence of negative numbers results in the presentation of several related methods such as "excess and deficits" and "double excess" to solve a system of 2 conditions with 2 unknowns [**3**]. In modern terms, we would describe these problems similarly, but with positive or negative results. In this section, we will first introduce the "Excess and Deficits" method in mordern notations, then tracing back to how ancient Chinese mathematicians have done it on the Counting Board.

## 3.1 Method for "Excess and Deficit" in modern notations [6]

Noting that the "Excess and Deficit" problem can be translated into a system of 2 conditions in 2 unknowns, we can denote such a system in the following way. $x_1$ and $x_2$ are the two unknowns, $e$, $(e > 0)$ stands for excess and $d$, $(d > 0)$ stands for deficit. $a_1$ is a constant that gives the result of excess, $a_2$ is a constant that gives the result of deficit. Hence, mathematically, we can get:

$$a_1 x_1 - x_2 = e \ (1)$$

$$a_2x_1 - x_2 = -d \ (2)$$

In other words, we can denote the equations into the matrix form:

$$\begin{bmatrix} a_1 & -1 \\ a_2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} e \\ -d \end{bmatrix}$$

from the two equations we defined, subtracting equation 1 with equation 2, we can get,

$$a_1x_1 - a_2x_1 = e + d \Rightarrow x_1 = \frac{e+d}{a_1 - a_2}$$

Then multiplying $a_1$ for equation (1), and multiplying $a_2$ for equation (2), subtracting equation 1 with equation 2, we can get

$$a_1x_2 - a_2x_2 = a_2e + a_1d \Rightarrow x_2 = \frac{a_2e + a_1d}{a_1 - a_2}$$

Referencing on the original text in *The Nine Chapters on the Mathematical Art* [**6**], the solution is given in division, so we can convert them as follows:

$$x_1 = (e + d) \div (a_1 - a_2)$$

$$x_1 = (a_2e + a_1d) \div (a_1 - a_2)$$

## 3.2 Method for "Excess and Deficit" in Counting Boards [6]

In general we will divide the process into 8 steps, matching to the snetences of instructions given in the book (On page 31 [7.4] [1]) and visualize the steps on the counting board.

1. Step 1: Put down the proposed values ("suo chu lü"), and below each, the corresponding surplus or deficit.

| | |
|---|---|
| $a_1$ | $a_2$ |
| $e$ | $d$ |

2. Step 2: Then cross multiply the contribution rates, and add them to become the upper quantity (shi).

| | |
|---|---|
| $a_1$ | $a_2$ |
| $e$ | $d$ |
| $a_1d + a_2e$ | |

3. Step 3: Combine the excess and deficit to become the lower quantity (fa).

| | |
|---|---|
| $a_1$ | $a_2$ |
| $e$ | $d$ |
| $a_1d + a_2e$ | |
| $d + e$ | |

4. Step 4: Again, place the rates of contribution on the board, subtract the lesser from the greater.
   Noting that this line is equivilent to let us finding the exact positive differences between $a_1$ and $a_2$, so that we can denote the remainder as $|a_1 - a_2|$.

| | |
|---|---|
| $a_1$ | $a_2$ |
| $e$ | $d$ |
| $a_1d + a_2e$ | |
| $d + e$ | |
| $\|a_1 - a_2\|$ | |

5. Step 5: Divide the shi and fa by the remainder. [Division of] shi gives the cost of the object and [Division of] fa gives the number of persons.

| | |
|---|---|
| $a_1$ | $a_2$ |
| $e$ | $d$ |
| $a_1d + a_2e \div \|a_1 - a_2\|$ | |
| $d + e \div \|a_1 - a_2\|$ | |
| $\|a_1 - a_2\|$ | |

The formulation of the "Excess and Dificit" problem on counting board as a rectangular represents as the precursor of the matrix definition from ancient Chinese mathematicians. In general, the systems of linear algebra in ancient China was built up upon on such foundation but dealing with more complicated real life and abstract problems.

# 4 The "Fangcheng" Procedure

In this chapter, *The Nine Chapters on the Mathematical Art* elucidates mainly three features that focusing on modern linear algebra:
For higher-order polynomial systems, the general "FangCheng" problems are arranged on a 2D counting board, illustrating the ancient method's spatial approach to mathematical equations. The initial step in this procedure, akin to what is known in modern mathematics as **forward elimination**, involves using one equation, starting from the first, to eliminate variables from subsequent equations. This manipulation results in the equations forming a structure that resembles a triangular array. Interestingly [**3**], the process of elimination was closely paralleled to Gaussian Elimination, utilizing a technique termed "*bian cheng*," or "*cross multiplication*." This specific method entails multiplying an entire column by a single entry to streamline the system of equations.
Furthermore, the FangCheng method's approach to back substitution distinctively diverges from contemporary linear algebra's more intuitive methods. This part of the discussion will focus on analyzing and contrasting these methodologies, highlighting the unique aspects of the FangCheng process and its historical significance in the evolution of algebraic thought.

## 4.1 Matrix Formulation

The Chinese array, as placed on the counting board, is similar to the corresponding augmented matrix in modern linear algebra, except for the system for writing numbers, and the orientation of the matrix. [**1**] (Noting for ancient Chinese mathematicians, they use *counting rods* to replace the variables listed in here.)

| | | | | |
|---|---|---|---|---|
| $a_{n,1}$ | $a_{n-1,1}$ | $\cdots$ | $a_{2,1}$ | $a_{1,1}$ |
| $a_{n,2}$ | $a_{n-1,2}$ | $\cdots$ | $a_{2,2}$ | $a_{1,2}$ |
| $a_{n,n}$ | $a_{n-1,n}$ | $\cdots$ | $a_{2,n}$ | $a_{1,n}$ |
| $b_n$ | $b_{n-1}$ | $\cdots$ | $b_{2,n}$ | $a_{1,n}$ |

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & b_n \end{bmatrix}$$

## 4.2 Elimintation [7]

Referencing problem 1 on the original text in *Nine Chapters* and Liu Hui's commentary [**3**][**6**], here is the general routine of how elimination proceed. For simplicity, this article will use modern notation to represent values in the rectangular array.
The initial placement of counting rods for problem 1 in augmented matrix form is as follows:

$$\begin{bmatrix} 3 & 2 & 1 & | & 39 \\ 2 & 3 & 1 & | & 34 \\ 1 & 2 & 3 & | & 26 \end{bmatrix}$$

The second column of counting rods is multiplied by 3, yielding $(6, 9, 3, 102)$. Then, the first column [row] is subtracted twice from the second row, yielding $(0, 5, 1, 24)$.

$$\begin{bmatrix} 3 & 2 & 1 & | & 39 \\ 0 & 5 & 1 & | & 24 \\ 1 & 2 & 3 & | & 26 \end{bmatrix}$$

Then, multiply the third row by 3, yielding $(3, 6, 9, 78)$, subtracting the first row, getting $(0, 4, 8, 39)$.

$$\begin{bmatrix} 3 & 2 & 1 & | & 39 \\ 0 & 5 & 1 & | & 24 \\ 0 & 4 & 8 & | & 39 \end{bmatrix}$$

Lastly, we replicate the similar routine by multiply the third row by 5, and subtract the second row with 4 times, reverting the rectangular array into upper triangular form.

$$\begin{bmatrix} 3 & 2 & 1 & | & 39 \\ 0 & 5 & 1 & | & 24 \\ 0 & 0 & 36 & | & 99 \end{bmatrix}$$

## 4.3 Back substitution using the Modern Approach [7]

In the conventional modern approach, the augmented matrix in represents the following equations. [5]

$$3x_1 + 2x_2 + x_3 = 395,\ 5x_2 + x_3 = 24,\ 36x_3 = 99$$

- Calculate the value of $x_3$ using equation, yielding:

$$x_3 = \frac{99}{36} = \frac{11}{4}$$

- Then we subsitute $x_3$ into the equation 2, getting:

$$5x_2 - \frac{11}{4} = 24 \quad \text{yielding} \quad x_2 = \frac{17}{4}$$

- Last, subsitute $x_3$ and $x_2$ into equation 1, having:

$$3x_1 + 2 \times \frac{17}{4} + \frac{11}{4} = 39 \quad \text{yielding} \quad x_1 = \frac{37}{4}$$

Apart from the modern approach, which is logical and easy to follow numerically, In the *Fangcheng* procedure, the quantities used for back substitution are not correspond to any easily conceptualizable quantity any more.

## 4.4 Back substitution using the Fancheng Procedure [6]

In the *Fangcheng* procedure, the ancient Chinese mathematicians are try to avoid fraction addition and multiplication. Hence, they take a different approach:
For calculating the third unknown, instead of getting $\frac{99}{36}$ we keep using 99 in our back substitution. the value 99 represents $36x_3$ but nothing specific since 36 is nothing more than the result derived from the elimination.
To calculate the second unknown, we then "cross multiply," subtract and divide in the following manner:

$$(36 \times 24 - 99) \div 5 = 153$$

Then we again "cross multiply," subtract and divide:

$$(39 \times 36 - 2 \times 153 - 99) \div 3 = 333$$

Lastly, we can derive all three unknowns by dividing all three results overall:

$$x_1 = 333 \div 36 = \frac{37}{4}, \quad x_2 = 153 \div 36 = \frac{17}{4} = 38, \quad x_3 = 99 \div 36 = \frac{11}{4}$$

### 4.5 Visualization on Counting board

Furthermore, we can visualized the back substitution procedure on the counting board:
"Cross multiply" by $b_2 = 24$ by the final pivot $a_{33} = 36$; then multiply $a_{23} = 1$ by the constant term $b_3 = 99$.

$$\begin{bmatrix} 3 & 2 & 1 & 39 \\ 0 & 5 & 1 & 24 \\ 0 & 0 & 36 & 99 \end{bmatrix} \Rightarrow \begin{bmatrix} 3 & 2 & 1 & 39 \\ 0 & 5 & 99 & 864 \\ 0 & 0 & 36 & 99 \end{bmatrix}$$

We "eliminate" the entries in the second row, by subtracting 99 from 864, then dividing by 5, giving $(864 - 99)/5 = 153$.

$$\begin{bmatrix} 3 & 2 & 1 & 39 \\ 0 & 0 & 0 & 153 \\ 0 & 0 & 36 & 99 \end{bmatrix}$$

"Cross multiply" by $b_1 = 39$, by the final pivot $a_{33} = 36$; then multiply $a_{13} = 1$ by the constant term $b_3 = 99$ and $a_{12} = 2$ by $b_2 = 153$.

$$\begin{bmatrix} 3 & 2 & 1 & 39 \\ 0 & 0 & 0 & 153 \\ 0 & 0 & 36 & 99 \end{bmatrix} \Rightarrow \begin{bmatrix} 3 & 306 & 99 & 1404 \\ 0 & 5 & 99 & 153 \\ 0 & 0 & 36 & 99 \end{bmatrix}$$

Again, we "eliminate" the entries in the final row by subtracting 99 and 306 from 1404 and dividing by 3, $(1404 - 99 - 306)3 = 333$. Then divide 36 into the constant terms.

$$\begin{bmatrix} 0 & 0 & 0 & 333 \\ 0 & 0 & 0 & 153 \\ 0 & 0 & 36 & 99 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 0 & 0 & \frac{333}{36} \\ 0 & 0 & 0 & \frac{153}{36} \\ 0 & 0 & 0 & \frac{99}{36} \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 0 & 0 & \frac{37}{4} \\ 0 & 0 & 0 & \frac{17}{4} \\ 0 & 0 & 0 & \frac{11}{4} \end{bmatrix}$$

Overall, the *Fangcheng* procedure avoids fractions until the final step. As noting, the back substitution in the *Fangcheng* procedure for solving differs from the conventional modern approach.

## 5 The procedure for positives and negatives

Ancient Chinese mathematicians employed a sophisticated system using counting rods to handle calculations involving positives and negatives. They designated red rods for positive values and black rods for negative ones. When colors were not available, rods were placed in slanted or upright positions as an alternative method to distinguish between positive and negative. These rods, when organized into columns, facilitated mathematical operations: rods of the same color in a single column represented addition, whereas rods of different colors indicated subtraction. The position of rods within a column was also crucial, as it represented different numerical values or variables. To ensure accuracy and clarity in the documentation and application of these methods, clear distinctions between positive and negative rods were maintained. Moreover, the process of comparing and adjusting rods helped in setting boundaries for calculations, distinguishing between increments and decrements, and aligning dissimilar values by modifying signs or employing constants, thus preserving the integrity of numerical values throughout the calculation process.

# 6 Algorithmic Analogy

The procedure of *Guassian Elimination* recorded in the *The Nine Chapter of Mathematical Arts* is exactly identical to what proceed in today's math in general. However, the Ancient Chinese mathematicians did not account for the corner case when the given matrix $A$ is signular. For *Back Subsitution*, the routine quite different to what we've used nowadays. The following is the formal pseudocode algorithm representating *Guassian Elimination* and *Back Subsitution*:

**Algorithm 1** Elimination

```
procedure Elimination(A, b)
    for k = 1 to n − 1 do
        if A[k, k] = 0 then
            found ← false
            for m = k + 1 to n do
                if A[m, k] ≠ 0 then
                    Swap row k with row m in A and b
                    found ← true
                    break
            if !found then
                error "Matrix is singular"
                return null
        else
            for i = k + 1 to n do
                a[i] ← A[i, k]
                for j = k + 1 to n do
                    A[i, j] ← A[k, k] · A[i, j]
                for j = k to n + 1 do
                    A[i, j] ← A[i, j] − a[i] · A[k, j]
    return A′, b′
```

**Algorithm 2** Back Substitution [**6**]

```
procedure Substitution(A, b, x)
    z[n] ← A[n, n] · A[n, n + 1]
    for k = n − 1 to 1 do
        z[k] ← A[k, n + 1]
        for i = n to k + 1 do
            z[k] ← z[k] − A[k, i] · z[i]
        z[k] ← z[k]/A[k, k]
    for k = n to 1 do
        x[k] ← z[k]/A[n, n]
    return x
```

# 7 Is *Fangcheng* procedure integer preserving?

For this section, we will have a superficial glance about how the Fangcheng procedure avoid the calculation with fractions in all general cases (Specifically during the process of "back subsitution").

## 7.1 Analysis of Results on the Row Reductions

As mentioned in the **Matrix Formulation** section, the *fangcheng* procedure normally starts with forming an agumented matrix.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} & b_n \end{bmatrix}$$

Then, we need to eliminate the non diagonal entries, transferring the matrix into upper triangular format [**6**]. The first step needed is to multiply the second row by the first entry left in the first row (Specifically: $a_{11}$), yielding,

$$a_1 1 \cdot (a_2 1, a_2 1, a_2 1, \ldots, a_2 n, b_2)$$

Afterwards, we multiply the first row by the first entry in the second row (Specifically: $a_{21}$), yielding,

$$a_2 1 \cdot (a_1 1, a_1 2, a_1 3, \ldots, a_1 n, b_1)$$

and subtract, giving the second row as,

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & b_1 \\ 0 & a_{11}a_{22} - a_{12}a_{21} & a_{11}a_{23} - a_{13}a_{21} & \cdots & a_{11}a_{2n} - a_{1n}a_{21} & a_{11}b_2 - b_1 a_{21} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} & b_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} & b_n \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & b_1 \\ 0 & \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} & \cdots & \begin{vmatrix} a_{11} & a_{1n} \\ a_{21} & a_{2n} \end{vmatrix} & \begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} & b_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} & b_n \end{bmatrix}$$

Noting that for all entries in the second row except the 0 term, we can use determinat symbols to get a more conicse representation.

### 7.1.1 Extension to General Approach

Starting from the previous gist as the foundation, now we can consider the *fangcheng* problems in a general viewpoint. In this artile, we will consider three special cases for the first set of row reductions in a matrix:

1. If $a_{21} = 0$, there is no subtraction, and the row remains unchanged. We multiply the second row by $\frac{1}{a_{11}}$ to achieve the correct result:

$$a_{2j}^{(1)} = \frac{1}{a_{11}} (a_{11}a_{2j} - 0 \cdot a_{1j}) = \frac{a_{11}a_{2j}}{a_{11}}.$$

2. If $a_{21} = a_{11}$, we subtract the first row from the second and correct by a factor of $\frac{1}{a_{11}}$:

$$a_{2j}^{(1)} = a_{2j} - a_{1j} = \frac{1}{a_{11}}\left(a_{11}a_{2j} - a_{21}a_{1j}\right) = \frac{a_{11}a_{2j} - a_{21}a_{1j}}{a_{11}}.$$

3. If $\gcd(a_{11}, a_{21}) \neq 1$, we multiply the second row by $\frac{a_{11}}{(a_{11},a_{21})}$ and subtract the first row multiplied by $\frac{a_{21}}{(a_{11},a_{21})}$:

$$a_{2j}^{(1)} = \frac{a_{11}}{(a_{11}, a_{21})}a_{2j} - \frac{a_{21}}{(a_{11}, a_{21})}a_{1j} = \frac{a_{11}a_{2j} - a_{21}a_{1j}}{(a_{11}, a_{21})}.$$

In practice, these special cases present complications when the greatest common divisor is not trivial, particularly for large numbers where determining the greatest common divisor may be challenging [**6**]. We define a constant $k_{2j}$ to adjust the $j$-th entry of the second row during matrix row reductions, tailored for the condition of the first two elements of the matrix. For the snippet of the part we selected, we will need to multiplying entires $\begin{vmatrix} a_{11} & a_{1j} \\ a_{21} & a_{2j} \end{vmatrix}$ by $\frac{1}{k_{21}}$. The correction factor $(k_{21})$ is set as follows:

$$k_{i1} = \begin{cases} 1 & \text{if } (a_{i1}, a_{i1}) = 1, \\ a_{11} & \text{if } a_{i1} = 0, \text{or } a_{i1} = \pm a_{i1} \\ d_i & \text{if } (a_{11}, a_{21}) \neq 1, d_i \text{ is a integer divides } (a_{11}, a_{21}) \end{cases}$$

Following the definition of correction factor, the updated $j$-th entry of the second row, $a_{2j}^{(1)}$, will need to be updated correspondingly:

$$a_{2j}^{(1)} = \begin{cases} \begin{matrix} a_{11} & a_{1j} \\ a_{21} & a_{2j} \end{matrix} = a_{11}a_{2j} - a_{21}a_{1j} & \text{if } (a_{11}, a_{21}) = 1, \\ \frac{1}{a_{11}}\begin{vmatrix} a_{11} & a_{1j} \\ 0 & a_{2j} \end{vmatrix} = a_{2j} & \text{if } a_{21} = 0, \\ \frac{1}{a_{11}}\begin{vmatrix} a_{11} & a_{1j} \\ a_{11} & a_{2j} \end{vmatrix} = a_{2j} + a_{1j} & \text{if } a_{21} = \pm a_{11}, \\ \frac{1}{d}\begin{vmatrix} a_{11} & a_{1j} \\ a_{21} & a_{2j} \end{vmatrix} = \frac{a_{11}}{d}a_{2j} - \frac{a_{21}}{d}a_{1j} & \text{if } \gcd(a_{11}, a_{21}) \neq 1, \text{ where } d \mid \gcd(a_{11}, a_{21}). \end{cases}$$

The variable $d$ should be chosen to simplify the matrix as much as possible, which might involve the application of advanced number theoretic algorithms in practical scenarios.
Then, we repeat the defined process, eliminating the column in succession. In general, each row is multiplied by the first element of the first row, $a_{11}$, and this product is subtracted from the product of the first row and the first element of the current row, $a_{i1}$. Also, we need to consider alternative approaches and therefore needed to introduce a unique constant for each row $i$, denoted $k_{i1}$.
Then, we can rewrite the result after the first row reduction as:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & b_1 \\ 0 & \frac{1}{k_{21}}\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \frac{1}{k_{21}}\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} & \cdots & \frac{1}{k_{21}}\begin{vmatrix} a_{11} & a_{1n} \\ a_{21} & a_{2n} \end{vmatrix} & \frac{1}{k_{21}}\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix} \\ 0 & \frac{1}{k_{31}}\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} & \frac{1}{k_{31}}\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} & \cdots & \frac{1}{k_{31}}\begin{vmatrix} a_{11} & a_{1n} \\ a_{31} & a_{3n} \end{vmatrix} & \frac{1}{k_{31}}\begin{vmatrix} a_{11} & b_1 \\ a_{31} & b_3 \end{vmatrix} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \frac{1}{k_{41}}\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \frac{1}{k_{41}}\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} & \cdots & \frac{1}{k_{41}}\begin{vmatrix} a_{11} & a_{1n} \\ a_{21} & a_{2n} \end{vmatrix} & \frac{1}{k_{41}}\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix} \end{bmatrix}$$

For simplicity purposes, we can simplify this matrix using the define $a_{ij}^{(1)}$ notation as:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & b_1 \\ 0 & a_{22}^{(1)} & a_{23}^{(1)} & \cdots & a_{2n}^{(1)} & b_2^{(1)} \\ 0 & a_{32}^{(1)} & a_{33}^{(1)} & \cdots & a_{3n}^{(1)} & b_3^{(1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & a_{n2}^{(1)} & a_{n3}^{(1)} & \cdots & a_{nn}^{(1)} & b_n^{(1)} \end{bmatrix}$$

Then we keep moving on, eliminate the second entries in row 3 to $n$, recurring the previous definition and based on the previous calcuated matrix. Hence, we can denote the new correction factor $k_{i2}$ as:

$$k_{i2} = \begin{cases} 1 & \text{if } (a_{22}^{(1)}, a_{i2}^{(1)}) = 1, \\ a_{22}^{(1)} & \text{if } a_{i2}^{(1)} = 0, \text{or } a_{i2}^{(1)} = \pm a_{i2}^{(1)} \\ d_i & \text{if } (a_{22}^{(1)}, a_{i2}^{(1)}) \neq 1, d_i \text{ is a integer divides } (a_{22}^{(1)}, a_{i1}^{(1)}) \end{cases}$$

Again, $d$ isn not necessarily $(a_{22}^{(1)}, a_{i1}^{(1)})$. Thus similarly, we have to multiply the $i^{th}$ row by

$$\frac{1}{k_{i2}} a_{22}^{(1)} = \frac{1}{k_{i2}} \frac{1}{k_{21}} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

Then, subtract from the second row multiply by,

$$\frac{1}{k_{i2}} a_{i2}^{(1)} = \frac{1}{k_{i2}} \frac{1}{k_{i1}} \begin{vmatrix} a_{11} & a_{12} \\ a_{i1} & a_{i2} \end{vmatrix}$$

Hence, we can get the entries of $i^{th}$ become,

$$a_{ij}^{(2)} = \frac{1}{k_{i2}} \frac{1}{k_{21}} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \cdot \frac{1}{k_{21}} \begin{vmatrix} a_{11} & a_{1j} \\ a_{i1} & a_{ij} \end{vmatrix} - \frac{1}{k_{i2}} \frac{1}{k_{i1}} \begin{vmatrix} a_{11} & a_{12} \\ a_{i1} & a_{i2} \end{vmatrix} \cdot \frac{1}{k_{21}} \begin{vmatrix} a_{11} & a_{1j} \\ a_{21} & a_{2j} \end{vmatrix}$$

$$\Leftrightarrow a_{ij}^{(2)} = \frac{a_{11}}{k_{21} k_{i1} k_{i2}} \begin{vmatrix} a_{11} & a_{12} & a_{1j} \\ a_{21} & a_{22} & a_{2j} \\ a_{i1} & a_{i2} & a_{ij} \end{vmatrix}$$

Subsituting the simplication after the reduction, we can get the overall matrix representationa as follow:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & b_1 \\ 0 & \frac{1}{k_{21}} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \frac{1}{k_{21}} \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} & \cdots & \frac{1}{k_{21}} \begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix} \\ 0 & 0 & \frac{a_{11}}{k_{21} j_{31} k_{32}} \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} & \cdots & \frac{a_{11}}{k_{21} j_{31} k_{32}} \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \frac{a_{11}}{k_{21} j_{31} k_{32}} \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{n1} & a_{n2} & a_{n3} \end{vmatrix} & \cdots & \frac{a_{11}}{k_{21} j_{31} k_{32}} \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{n1} & a_{n2} & b_n \end{vmatrix} \end{bmatrix}$$

## 7.2 Supplemental Notations

Now, for the simplicity of the following calculation, it is significant to introduce several notations [**6**]. Let $\{i_1, i_2, \cdots, i_r\}$ and $\{j_1, j_2, \cdots, i_r\}$ be subsequence of $\{1, \cdots, n\}$ with $r$ elements. Then the minor determinant of order $r$ can be denoted as:

$$\begin{bmatrix} a_{i_1,j_1} & a_{i_1,j_2} & \dots & a_{i_1,j_r} \\ a_{i_2,j_1} & a_{i_2,j_2} & \dots & a_{i_2,j_r} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i_r,j_1} & a_{i_r,j_2} & \dots & a_{i_r,j_r} \end{bmatrix}$$

Which lies as in the intersection of rows and columns $\{i_1, i_2, \cdots, i_r\}$ and $\{j_1, j_2, \cdots, i_r\}$.

## 7.3 Calculating $a_{ij}^{(r)}$

Using the previous annotated notation, and the *Schur Complement*, we can write $a_{ij}^{(r)}$ as following:

$$a_{ij}^{(r)} = \left[\frac{A\begin{bmatrix}1\\1\end{bmatrix}}{k_{21}}\right]^{2^{r-2}} \cdot \left[\frac{A\begin{bmatrix}1,2\\1,2\end{bmatrix}}{k_{31}k_{32}}\right]^{2^{r-2}} \cdots \left[\frac{A\begin{bmatrix}1,\cdots,r,i\\1,\cdots,r,j\end{bmatrix}}{k_{i1}k_{i2}\cdots k_{ir}}\right]$$

[1]

Then after completing series of $r-1$ row reduction, we can get the original matrix in a row reduction manner, namely:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1,n-1} & a_{1n} & b_1 \\ 0 & a_{22}^{(1)} & a_{23}^{(1)} & \dots & a_{2,n-1}^{(1)} & a_{2n}^{(1)} & b_2^{(1)} \\ 0 & 0 & a_{33}^{(2)} & \dots & a_{3,n-1}^{(2)} & a_{3n}^{(2)} & b_3^{(2)} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & a_{n-1,n-1}^{(n-2)} & a_{n-1,n}^{(n-2)} & b_{n-1}^{(n-2)} \\ 0 & 0 & 0 & \dots & 0 & a_{nn}^{(n-1)} & b_n^{(n-1)} \end{bmatrix}$$

[1]

**Theorem 7.1.** *Lemma C.1 (Schur Complement). Let $A$ be an $n \times n$ matrix with $\det A \neq 0$. For $r$ such that $1 \leq r < n$, partition $A$ into blocks such that $A_{11}$ is the leading principal submatrix of order $r$ and $A_{22}$ is the trailing principal submatrix of order $n-r$, so that*

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}.$$

If $\det A_{11} \neq 0$, then

$$\det A = \det A_{11} \det(A_{22} - A_{21}A_{11}^{-1}A_{12}),$$

and $A_{22} - A_{21}A_{11}^{-1}A_{12}$ is called the Schur complement of $A_{11}$ in $A$.

### 7.4 Calculating $a_{nn}^{(n-1)}$

Following with the assumption listed above, this is just a extension for the previous equation focusing on the last term on the last row in row reduced form of the original matrix.

$$a_{nn}^{(n-1)} = \left[\frac{A\begin{bmatrix}1\\1\end{bmatrix}}{k_{21}}\right]^{2^{n-3}} \cdot \left[\frac{A\begin{bmatrix}1,2\\1,2\end{bmatrix}}{k_{31}k_{32}}\right]^{2^{n-4}} \cdots \left[\frac{A\begin{bmatrix}1,\cdots,n\\1,\cdots,n\end{bmatrix}}{k_{n1}k_{n2}\cdots k_{n,n-1}}\right]$$

### 7.5 Calculating $z_j$

Now, by Cramer's Rule,

$$x_j = \frac{\det B_j}{\det A}$$

and since by definition, $A\begin{bmatrix}1,\cdots,n\\1,\cdots,n\end{bmatrix} = \det(A)$, we can derive the following equation:

$$z_j = a_{nn}^{n-1}x_j = \left[\frac{A\begin{bmatrix}1\\1\end{bmatrix}}{k_{21}}\right]^{2^{n-3}} \cdot \left[\frac{A\begin{bmatrix}1,2\\1,2\end{bmatrix}}{k_{31}k_{32}}\right]^{2^{n-4}} \cdots \left[\frac{A\begin{bmatrix}1,\cdots,n-2\\1,\cdots,n-2\end{bmatrix}}{k_{n_1,1}k_{n-1,2}\cdots k_{n-1,n-2}}\right]\left[\frac{\det B_j}{k_{n1}k_{n2}\cdots k_{n,n-1}}\right]$$

Thus we have the subsequent result that if at each step $r$ of the row reductions if $k_{ij} = 1, \forall i, j,\ (j < i \leq n)$ we multiply each row by $a_{nn}^{(n-1)}$, we can get:

$$z_j = a_{nn}^{n-1}x_j = a_{11}^{2^{n-3}}\left[A\begin{bmatrix}1,2\\1,2\end{bmatrix}\right]^{2^{n-4}}\left[A\begin{bmatrix}1,2,3\\1,2,3\end{bmatrix}\right]^{2^{n-5}} \cdots \left[A\begin{bmatrix}1,\cdots,n-2\\1,\cdots,n-2\end{bmatrix}B_j\right]$$

is always an integer, for all $j, (1 \leq j \leq)$. Since cross multiplication at every stage according to the *Fangcheng* precedure, if at no stage we use a reduced multipler, then all the $z_j$ are integers. Hence, this indicates that our final answer if fraction-free [**6**].

### 7.6 Rewritting $a_{nn}^{(n-1)}$ in terms of $a_{rr}^{(r-1)}$

Noticing that to previous calculation makes several assumptions that are not always true (We won't list them out specifically for conciseness), hence we can obtain a value that might be more accurately reflects the mathematical practice.
[2]

[2]

**Lemma 1** (Chio's Pivotal Condensation)**.** *If* $a_{11} \neq 0$*,*

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = a_{11}^{n-2} \times \begin{vmatrix} \frac{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}{a_{11}} & \cdots & \frac{\begin{vmatrix} a_{11} & a_{1n} \\ a_{21} & a_{2n} \end{vmatrix}}{a_{11}} \\ \vdots & \ddots & \vdots \\ \frac{\begin{vmatrix} a_{11} & a_{12} \\ a_{n1} & a_{n2} \end{vmatrix}}{a_{11}} & \cdots & \frac{\begin{vmatrix} a_{11} & a_{1n} \\ a_{n1} & a_{nn} \end{vmatrix}}{a_{11}} \end{vmatrix}.$$

for $n \geq 3$, we can write $z_j = a_{nn}^{n-1} x_j$ as follows (Using Chio's Pivotal Condensation):

$$z_j = a_{nn}^{(n-1)} x_j = \left( \prod_{r=1}^{n-2} \left( \frac{\left(a_{rr}^{(r-1)}\right)^{n-r-1}}{\prod_{i=r+1}^{n} k_{ir}} \right) \right) \frac{\det B_j}{k_{n,n-1}}$$

For each of the term on $RHS$, with the exception of the first element, $a_{nn}^{(n-1)}$, divided by $n-r$ terms $k_{ir}$. By definition, each $k_{ir}$ divides $a_{rr}^{(r-1)}$, so each grouping will be:

$$\frac{(a_{rr}^{(r-1)})^{n-r-1}}{k_{r+1,r} k_{r+2,r} \cdots k_{n-1,r} k_{n,r}}$$

is integer if at least one of the $k_{ir} = 1$. Hence we can state sufficiently that each grouping will be an integer if:

$$k_{ir} = 1 \text{ for at least on } i \text{ with } (r+1 \leq i \leq n)$$

This is equivilent stating a sufficient conidtion that each $z_j$ is an integer is that for each $r = 1, \ldots, n-1$, $k_{ir} = 1$ for at least one $i$ with $(r+1 \leq i \leq n)$. In other words, in the *fangcheng* method, each $z_j$ is guaranteed to be an integer if every pivot $a_{rr}^{(r-1)}$ is utilized in at least one operation to nullify subsequent row entries [**6**]. This means that fractional numbers are typically not present in the *fangcheng* procedure's computations.

# 8 Summary

In this paper, we briefly went through "Excess and Deficit" problem from both mordern and ancient perspectives two related problems examined. Moreover, we shifts our focus to "The Fangcheng procedure" elevate viewpoint from systems of equations with two unkowns to higher order linear equations. We also formulated two pseudocodes for the "elimination" and "back subsitution", viewing the their essences from ancient to modern. Laslty, we examined the general routine of how ancient Chinese deal with postivies and negatives in rectangular array and discussed the Integer Preserving Property in a general glance.

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